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## STRING THEORY AND THE FUZZY TORUS

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We outline a brief description of non commutative geometry and present some applications in string theory. We use the fuzzy torus as our guiding example.

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### 1. Introduction

Geometry, the science of properties of spaces, has been, along the centuries, referring to different definitions of its domain of interest. As a general fact, the tendency has been to widen the scope of geometry and the concept of "admissible spaces", along with an abstraction process that shifted the focus from the properties of spaces (as experienced by an "external" observer living in some embedding geometrical entity, or by an "intrinsic" observer living in the space itself) to the relations among entities belonging to the space ("geometrical objects"). At the same time, the features of spaces have evolved from being very concrete and almost physical, to a level of abstraction which essentially rephrases them in analytical terms. Geometry did not certainly become less interesting along this path, which has also been giving birth to many interesting and fruitful branches of mathematics - and noncommutative geometry is just one of them. We will here aim to sketch a brief outline of non commutative geometry and its possible applications, and discuss some examples of string theory models which enjoy a non commutative description.

### 2. Non-commutative geometry

Both physics and mathematics, and in particular geometry which specifically is devoted to the study of spaces, often deal with the task of investigating objects whose direct analysis is overly difficult, exceedingly cumbersome or impossible tout court.

Let us consider, for example, a topological space. Given a generic set  $X$  endowed with a topology,  $\mathcal{M}$ , handling directly its topology would involve not only heavy operations over sets that we are not familiar with, but would also require the

handling of an enormous number of subsets — they may be easily as many as  $\mathcal{P}(X)$ . So such an approach is reserved to the situation in which either the topology is naturally “easy” — but, often, uninteresting — or when it is possible to isolate a basis for the topology, that is if we can completely derive it out of a small number of open sets.

The standard procedure to study a topological space is to realize that it is the natural habitat to grow the notion of a continuous function. We consider the function  $f : X \rightarrow \mathbb{R}$ , and exploit the ease to do computations with numbers and the fact that the numerical fields have an obvious notion of “neighbour” among the complexity of their structures. We will say that  $f$  is continuous if the counterimage of an open set of  $\mathbb{R}$  is still an open set for the topology  $\mathbb{M}$  chosen on  $X$ . And we can then proceed to analyze the class of continuous functions over  $X$ , which enjoys naturally the structures of a commutative algebra, because it inherits them from the structures of real numbers. It is enough to define the sum and the product of functions as the function whose value at each point is the sum or product of the numerical values of the functions on each point:  $(f+g)(x) := f(x) + g(x)$ ,  $(f \cdot g)(x) := f(x) \cdot g(x)$ . Since this is obtained from the addition and multiplication of  $\mathbb{R}$  or  $\mathbb{C}$  only, we remark that the same construction can be carried out for, as an example, matrix valued continuous functions, the only difference being the loss of commutativity.

The idea of studying a topological space through the algebra of its continuous functions is the central idea of algebraic topology. Likewise, algebraic geometry studies characteristics of spaces by means, for example, of their algebra of rational functions. In this framework it is very natural to ask what happens if we replace the algebra of ordinary functions with some non-commutative analogue, both to extend the tools to new and previously intractable contexts (and sometimes this replacement is unavoidable) and to study with more refined probes the “classical spaces” (which leads sometimes to new and surprising results).

### 2.1. *Quotient spaces*

A class of typically intractable (and typically very interesting) objects is reached by means of a quotient space construction, that is, by considering a set endowed with an equivalence relation fulfilling reflexivity, symmetry and transitivity axioms and identifying the elements which are equivalent with respect to such a relation. In the following we will find particularly useful the “graph” picture of an equivalence relation: if we consider the Cartesian product of two copies of the set, we can assign the equivalence relation as a subset of the Cartesian product (the one formed by the couples satisfying the relation). The construction gives interesting results already if we consider a set (possibly with an operation), but of course is much richer if we act over a set with structure.

Let’s clarify our statement with an example<sup>1</sup>. Consider the flat square torus, that is,  $[0, 1] \times [0, 1]$  with the opposite sides ordinally glued. This is clearly a well-

behaved space and, undoubtedly, a compact one<sup>a</sup>. Let's introduce an equivalence relation which identifies the points of the lines parallel to  $y = \sqrt{2}x$ , that is, we “foliate” the space into leaves parametrized by the intercept. Since  $\sqrt{2}$  is irrational, though, any such leaf fills the torus in a dense way (that is, given a leaf and a point of the torus, the leaf is found to be arbitrarily close to the point). If we try to study the quotient space and to introduce in it a topology, we will find that “anything is close to anything”, that is, the only possible topology contains as open sets only the whole space and the empty set. It is hopeless to try to give the quotient space an interesting topology based on our notion of “neighborhood” of the parent space.

It is, in particular, hopeless for all practical purposes to give the space the standard notion of topology inherited by the quotient operation, which we shortly describe. If we have a space  $A \equiv B / \sim$ , there is a natural projection map

$$\begin{aligned} p : B &\longrightarrow A \\ p : x &\longmapsto [x] \end{aligned} \tag{1}$$

which sends  $x \in B$  in its equivalence class. The inherited topology on  $A$  would be the one whose open sets are the sets whose counterimages are open sets in  $B$ .

We want an interesting topology, richer than  $\{\emptyset, X\}$ , and, moreover, we would like the topological space so obtained to enjoy local compactness. The reason why we make the effort is that the dull topology  $\{\emptyset, X\}$  treats the space, from the point of view of continuous functions which will be our probe, as the space consisting of only one point; all the possible subtleties of our environment will be lost. Local compactness is a slightly more technical tool, but we can imagine, both from the physical and the mathematical point of view, why it is so useful. Each time we have a nontrivial bundle (and we will have plenty of them in the following) we usually define them not globally, but on neighbourhoods. Since these “patches” will in general intersect, we need a machinery to enforce agreement of alternative descriptions. Local compactness (and similar tools) ensure us that “the number of possible alternative descriptions will never get out of control”. We shall see how to achieve the notable result of introducing in “weird” spaces a rich enough and even locally compact topology. An example of the process is presented in the next section.

## 2.2. A typical example: the space of Penrose tilings

We are going to discuss a situation which embodies most of the characteristic features both of the problems which non-commutative geometry makes tractable and of the procedure which allows their handling<sup>2</sup>.

<sup>a</sup>We say that a topological space with a given topology is compact if any open covering of the space (i. e. any family of open sets whose union is the space itself) admits a finite subcovering (i. e. a finite subfamily of the above which is still a covering). We say that a topological space  $(X, \mathcal{U})$  is locally compact if for all  $x \in X$  and for all  $U \in \mathcal{U}$ ,  $x \in U$ , there exist a compact set  $W$  such that  $W \subset U$ . It is, of course, useful to refer to a compactness notion also when we have more structures than just the one of a topological space (for example, differentiability).

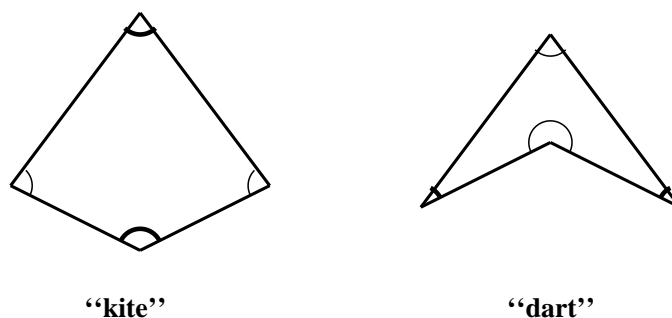


Fig. 1. Dart and Kites.

Penrose was able to build tilings of the plane having a 5-fold symmetry axis; this is not possible by means of periodic tilings with all equal tiles, as it is known since a long time. They are composed (see Fig. 1) of two types of tiles: “darts” and “kites”, with the condition that every vertex has matching colors. A striking characteristic of the Penrose tilings is that any finite pattern occurs (infinitely many times, by the way) in any other Penrose tiling. So, if we call identical two tilings which are carried into each other by an isometry of the plane (this is a sensible definition since none of the tilings is periodic), it is never possible to decide locally which tiling is which. However the distinct Penrose tilings are an uncountable infinity.

Yet one can prove that there exist numbers (that is integer numbers) that label (some of) the subtly different elements of the space — and it would be quite unpleasant to dismiss such a fact. After having shown that the space of distinct Penrose tilings can be described as  $X = K/\mathcal{R}$ , where  $K$  is compact and  $\mathcal{R}$  is some equivalence relation<sup>b</sup>, the path of non-commutative geometry is precisely to show (1) that the attempt of distinguishing the tilings by means of an algebra of operator-valued functions is successful, and (2) that it is actually sensible to say that there are different Penrose tilings, since topological invariants can be built and used to label the tilings. For a very elementary summary, see <sup>3</sup>.

### 3. Yang-Mills Theory on the Fuzzy Torus

In the next two sections we will concentrate on the simplest, yet physically non-trivial, example of a non-commutative geometry: the fuzzy torus. We will define the notion of a bundle over such geometry, which leads to what physicists call a

<sup>b</sup>This point could be quite confusing to the careful reader: since the  $\mathcal{R}$  equivalence classes contain a denumerable infinity of elements, while  $K$  has the cardinality of continuum, obviously the quotient space has more than one (and actually an awful lot of) elements. What we want to do is not a counting, but to make sure that the tilings are different in an *interesting* sense.

gauge theory. We will also explain how string theory provides a setting in which such structures arise naturally<sup>4,5,6,7,8,9,10</sup>.

### 3.1. Foliated torus and the non-commuting $U, V$ algebra

We would first like to clarify the relation between the so-called non-commuting torus geometry and the picture of the torus foliation, that is required in order to describe the non-commutative torus as a quotient, as outlined in the previous section. We will consider the situation where the slope of the leaf is irrational (that is, the leaf is infinite). Let us recall the main features of the two approaches.

In the non-commuting torus representation, one generalizes the toroidal geometry by supposing that the two coordinates are not ordinary variables, but satisfy the commutation relations

$$[p, q] = i\theta \quad (2)$$

This equation cannot of course be satisfied by finite dimensional matrices (no two finite dimensional matrices can have a commutator with a non-vanishing trace), but can be satisfied by a pair of operators,  $q$  and  $p$ , on a Hilbert space. A more convenient labeling in order to provide an useful representation in terms of finite dimensional matrices is

$$U = e^{ip}, \quad V = e^{iq} \quad \Rightarrow \quad UV = e^{i\theta} VU \quad (3)$$

In the case  $\theta = 2\pi/N$ , one can write down explicit  $N \times N$  matrices that satisfy the above relation, and will turn out to be very useful in the following. Let's have as  $U$  the shift matrix

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (4)$$

and as  $V$  the matrix of phases

$$V = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2\pi i/N} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{4\pi i/N} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{6\pi i/N} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{8\pi i/N} & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{pmatrix} \quad (5)$$

It is straightforward to check that (4), (5) satisfy

$$U^\dagger U = V^\dagger V = 1$$

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$$\begin{aligned} U^N &= V^N = 1 \\ UV &= VUe^{i\theta} \end{aligned} \quad (6)$$

with  $\theta = 2\pi/N$ .

Functions on the fuzzy torus are defined by the non-commutative analogue of a Fourier series: for any  $N \times N$  matrix  $\phi(U, V)$ , the expansion

$$\phi(U, V) = \sum_{n,m=0}^{N-1} c_{mn} U^n V^m \quad (7)$$

associates to it the function of two periodic variables whose Fourier modes are  $c_{n,m}$ . It is convenient to define

$$\begin{aligned} U_{mn} &= e^{-imn\theta} U^m V^n \\ \phi_{mn} &= e^{imn\theta} c_{mn} \\ \phi(U, V) &= \sum_{n,m=0}^{N-1} \phi_{mn} U_{mn} \end{aligned} \quad (8)$$

Note that the  $U_{mn}$  satisfy

$$U_{mn} U_{rs} = \exp \frac{1}{2} i\theta (ms - nr) \equiv U_{mn} * U_{rs} \quad (9)$$

Equation (9) defines the star-product on the fuzzy torus.

Let us switch to the foliated torus picture. Consider the square unit torus, that is,  $[0, 1] \times [0, 1]$  with the opposite sides ordinately glued. The foliation of  $\mathbb{R}^2$  generated by a family of lines of fixed slope:

$$y = ax + b \quad (a \text{ fixed}) \quad (10)$$

induces a corresponding foliation of the torus. Identifying the points belonging to any such leaf leads to the fuzzy torus.

A leaf of the foliation is parametrized by the value of  $b$ , but in a very redundant way. Actually two values of the intercept correspond to the same leaf provided that

$$b' = b + an \quad n \in \mathbb{Z} \quad (11)$$

Now let's introduce functions of two variables (one of which is an integer):  $F(b, n)$ . They actually admit to be interpreted as matrices (infinite, but with discrete entries), if one rewrites them in the form  $F(b, b')$  where  $b \sim b'$  (actually  $b' = b + an$ ). For such functions we are about to define a multiplication structure, which, of course, will be inspired by the usual matrix product:

$$H \equiv F * G \quad (12)$$

$$H(b, m) = \sum_n F(b, n) G(b + an, m - n) \quad (13)$$

The matrices of (13) are actually parametrized by  $b$ , but if we replace  $b$  with  $b' = b + ap$ , this results, at the level of the matrix product, in a relabeling (by

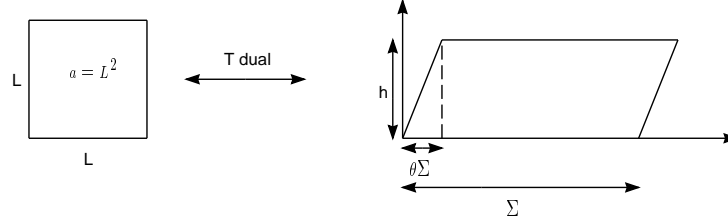


Fig. 2. T-duality.

positions) of rows and columns of an infinite matrix. In other words, they are actually dependent on the leaf only.

It is straightforward to check the matching of the  $*$  definition in (12-13) and of the matrix product in (9) and, thus, the equivalence of the two descriptions.

### 3.2. D-branes and non-commutative Yang-Mills theory

The main purpose of the present section is to connect the parameters of a super Yang-Mills theory on a fuzzy torus<sup>11</sup> (in particular the value of  $N$ ) with the geometrical viewpoint of the non-commutative foliation. The first step will be to just assume  $\theta = 1/N$ . We will later handle the more general case of rational non-commutative parameter:

$$\theta \equiv \frac{p}{N} \quad p, N \text{ relatively prime} \quad (14)$$

Following the approach of<sup>12,13</sup>, we consider a weakly coupled IIA theory<sup>c</sup> compactified on a square torus, with  $\tau = i$  and area  $a = \Sigma^{-1}h$ . We take a non zero value  $\theta$  for the background 2-form potential. It will be convenient (for T duality purposes) to choose as parameters of the system

- $\tau \equiv i$
- $P := \theta + i\Sigma^{-1}h$

Let's do now a T-duality transformation along one cycle of the torus, let's say, the "1" 1-cycle. Such a transformation interchanges  $\tau$  and  $P$ :

$$\begin{cases} \tau = i \\ P = \theta + i\Sigma^{-1}h \end{cases} \xleftrightarrow{\text{T duality}} \begin{cases} \tau' = \theta + i\Sigma^{-1}h \\ P' = i \end{cases} \quad (15)$$

Let the axis, corresponding to the "1" cycle along which T-duality is performed, be called  $\sigma$ , with  $0 \leq \sigma \leq \Sigma$ . We summarize the situation in Fig. 2.

What happens to a D0-brane under this duality? We will obtain a D1-brane oriented along the "1" direction of the new torus. (See Fig. 3.) Now suppose we

<sup>c</sup>We refer to Seiberg's<sup>14</sup> view of matrix theory<sup>15,16</sup>, that is, we replace the lightlike compactification with a compactification along a spacelike circle of shrinking radius.

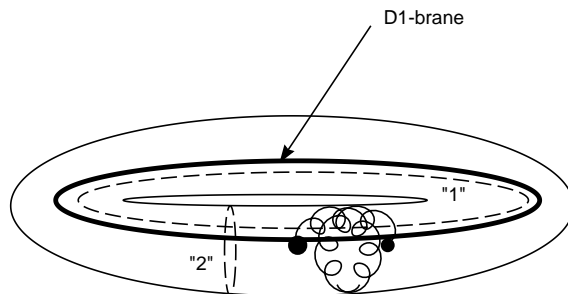
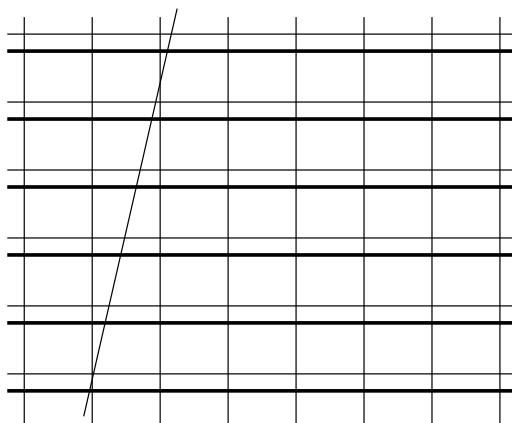
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Fig. 3. D1 Brane along the “1” direction.



(the line is to be imagined almost perpendicular)

Fig. 4. “Opened up” Torus.

have a fundamental string whose ends are on the D1-brane but which is wound  $w$  times along the “2” 1-cycle. If we try to impose a constraint of minimal length (provided the winding number  $w$  is fixed) and if we imagine to “open up” the torus and refer to a tiling of the plane made by its copies (see Fig. 4), we realize that the two points where the fundamental string is attached to the D-string are separated by a distance  $w\Sigma\theta$  along a straight line almost perpendicular to the D-string: they are equivalent points in the sense of the foliated torus construction described before.

Consider, at this stage, the (nonlocal) field operators which create and annihilate such strings. They are fields whose arguments are the two points,  $\sigma$  and  $\sigma + w\frac{\Sigma}{N}$  ( $w \in \mathbb{Z}$ ), of the D-string connected by the “minimal length” fundamental strings. As these two points are related by the equivalence relation of the foliated torus,

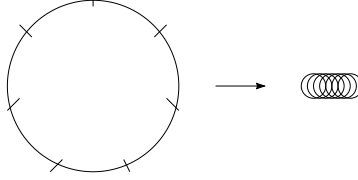


Fig. 5. Circle Folding.

it is natural to interpret such fields as objects which live in the non-commutative algebra of the foliated torus.:

$$\phi(\sigma, w) \quad (16)$$

To symmetrize the aspect of the above field, one might Fourier transform with respect to the periodic coordinate  $\sigma$ , thus obtaining a function of two integers:

$$\tilde{\phi}(k, w) \quad k, w \in \mathbb{Z} \quad (17)$$

The energy of such a mode is

$$E(k, w) = \left( \frac{1}{\Sigma^2} [k^2 + w^2] \right)^{\frac{1}{2}} \quad (18)$$

Let us now imagine of “folding” the circle according to the equivalence relation induced by the foliation (see Fig. 5). The small circle has now length  $\frac{\Sigma}{N}$  and its periodic coordinate  $x$  satisfies  $0 \leq x \leq \frac{\Sigma}{N}$ .

We should also notice that the choice of  $p$  in the numerator of  $\theta$  in (14) only corresponds to a relabeling of the  $N$  sectors, which are rearranged in a permuted order. (See Fig. 6). This is a consequence of the (trivial) fact that, given  $p$ ,  $N$  relatively prime, no one of the numbers

$$p, 2p, 3p, \dots (N-1)p$$

is a multiple of  $N$  and thus

$$\{0, (p)_{\text{mod } N}, (2p)_{\text{mod } N}, \dots\}$$

is a set of  $N$  integer numbers belonging to  $[0, N]$  and all different. Thus choosing  $p/N$  instead of  $1/N$  results only in a relabeling of the equivalence relation: the “arches” are superimposed in a different order.

We are now going to split the winding and the momentum modes into a part which is an integer multiple of  $N$  and the remainder (see Fig. 7). First we define an appropriate splitting of the integer  $k$  of equation (17):

$$k = KN + q \quad K, q \in \mathbb{Z} \quad 0 \leq q < N \quad (19)$$

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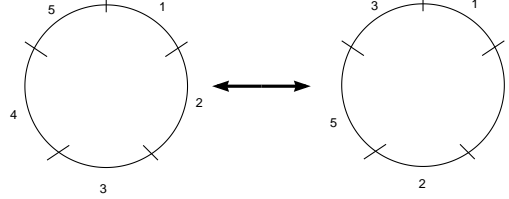


Fig. 6. Permutation.

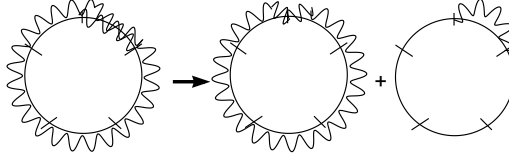


Fig. 7. Winding and Momentum Modes.

We simultaneously replace the fields  $\phi$  of eq. (16) with matrix valued fields

$$\phi_{ab}(x, W) \quad a, b = 1, \dots, N \quad (20)$$

where the index  $a$  (resp.  $b$ ) tells us in which of the  $N$  intervals (of length  $\frac{\Sigma}{N}$  each) the string begins (resp. ends) and the integer  $W$  is the winding number along the big circle of length  $\Sigma$ .

The relation with  $w$  is given by

$$w = NW + (a - b) \quad (21)$$

We can now rewrite the energy (18) in terms of the capital variables:

$$E^2 = \frac{1}{\Sigma^2} ((NK + q)^2 + (NW + (a - b))^2) \quad (22)$$

We now wish to reproduce the energy spectrum (22) with a description on a small torus of size  $\Sigma/N$ . On such torus we will have two directions,  $x$  and  $y$  respectively. The description (20) of the matrix fields can yield a description in one more direction if, as usual, we wish to interpret the winding mode as a Kaluza-Klein momentum in an additional direction; that is, Fourier transformation with respect to  $W$  gives

$$\phi_{ab}(x, y) \quad (23)$$

In order to reproduce the spectrum (22) we will introduce a background  $U(N)$  Yang-Mills field. We will also have to introduce non-trivial boundary conditions for the fields when transporting them around the  $x$  axis. The first couple of condition

we impose are

$$\phi_{a,b}(x + \frac{\Sigma}{N}, y) = \phi_{a+1,b+1}(x, y) \quad (24)$$

$$\phi_{ab}(x, y + \frac{\Sigma}{N}) = \phi_{ab}(x, y) \quad (25)$$

that is, if we move along  $x$  direction making a complete turn on the small circle we get a unit shift of both indexes (of the “begin” and “end” sectors on the big circle), while the  $y$  direction is associated to the “big” winding number. In matrix language

$$\phi(x + \frac{\Sigma}{N}, y) = U^\dagger \phi(x, y) U \quad (26)$$

$$\phi(x, y + \frac{\Sigma}{N}) = \phi(x, y) \quad (27)$$

where  $U$  is the  $N \times N$  shift matrix given in (4). Along the  $y$  direction we wish to introduce Wilson loops, in order to mimic the fractional contribution to the momentum which we encountered in eq. (22):

$$W(x) = \exp \left( i \oint A_y(x, y) dy \right) \quad (28)$$

We assume  $A_x = 0$  and  $A_y$  independent of  $y$ :

$$W(x) = \exp \left( i \frac{\Sigma}{N} A_y(x) \right) \quad (29)$$

We put in a background vector potential:

$$W(x) = \exp \left( i \frac{x}{\Sigma} \right) V \quad (30)$$

where the matrix  $V$  is as in (5). Notice that the background satisfies the same conditions of  $\phi$  if we make a complete turn on the small circle:

$$\Gamma(x + \frac{\Sigma}{N}) = U^\dagger \Gamma(x) U \quad (31)$$

From (29) and (30) it follows immediately for the vector potential

$$A_y = \frac{x}{\Sigma^2} N I + \frac{1}{\Sigma} \begin{pmatrix} \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot \end{pmatrix} \quad (32)$$

where  $I$  is the  $N \times N$  identity matrix.

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The vector potential can thus be split in an “abelian” and in a “non abelian” part. One should notice how the form of the abelian gauge field corresponds to a unit of abelian magnetic flux through the torus.

Now let’s consider the free part of the action for the scalars

$$\mathcal{L} = \int dx dy \text{Tr}(\dot{\phi}^2 - (\nabla\phi)^2) \quad (33)$$

where the covariant derivative is defined as usual

$$\nabla\phi = \partial\phi + i[A, \phi] \quad (34)$$

Let us consider separately the spectra of  $\nabla_x$  and  $\nabla_y$ .

First of all, the spectrum of  $\nabla_x \equiv \partial_x$  has to be evaluated on a circle of length  $\frac{\Sigma}{N}$ . But actually we imposed a condition which allows  $\phi$  to return to its value only after  $N$  turns, so the system behaves as the circle was effectively of size  $\Sigma$ . The spectrum will be then

$$\frac{i}{\Sigma}[KN + p] \quad p \in \mathbb{Z}, \quad 0 \leq p \leq N - 1 \quad (35)$$

with the same decomposition of eq. (19).

Second, let’s turn our attention to  $\nabla_y \equiv \partial_y + i[A_y, \cdot]$ . The derivative piece has the usual spectrum on the circle

$$i \frac{NW}{\Sigma} \quad W \in \mathbb{Z} \quad (36)$$

since  $\phi(y + \frac{N}{\Sigma}) = \phi(y)$ . The commutator (for which the “abelian” part of the vector potential can be dropped) is rewritten

$$[A_y, \phi]_{a,b} = \frac{1}{\Sigma}(a - b)\phi_{a,b} \quad (37)$$

and the spectrum of  $\nabla_y$  is

$$\frac{i}{\Sigma}[NW + (a - b)] \quad (38)$$

Since the equation of motion of  $\phi$  is

$$\ddot{\phi} = (\nabla_x^2 + \nabla_y^2)\phi \quad (39)$$

the spectrum will be

$$E^2 = \left[ \left( \frac{KN + p}{\Sigma} \right)^2 + \left( \frac{WN + (a - b)}{\Sigma} \right)^2 \right] \quad (40)$$

in agreement with eq. (22).

### 3.3. Interaction terms

We want to proceed to an explicit exhibition of how the star product works as a substitute of matrix multiplication and to discuss the role of the numerator of the fraction  $\theta \equiv p/N$ . To isolate our main point, we will refer to the quartic scalar interaction term, even if the whole procedure extends straightforwardly to the case of generic interactions.

In the 2+1 dimensional Connes-Douglas-Schwartz theory over the “big” torus, the form of the quartic terms is

$$\int_0^\Sigma dX dY [\phi^i * \phi^j - \phi^j * \phi^i]^2 \quad (41)$$

where  $X, Y$  are coordinates on the “large” torus and the star product is defined as

$$F * G = F(x, y) e^{i\frac{\theta}{2}(\bar{\partial}_x \bar{\partial}_y - \bar{\partial}_y \bar{\partial}_x)} G(x, y) \quad (42)$$

For the moment we will assume  $\theta \equiv 1/N$ .

To evaluate this, we go to the Douglas-Hull representation by Fourier transforming with respect to  $y$ :

$$\begin{aligned} F * G &= \sum_{n,m} F(x, n) e^{iny} e^{i\frac{\theta}{2}(\bar{\partial}_x \bar{\partial}_y - \bar{\partial}_y \bar{\partial}_x)} \tilde{G}(x, m) e^{imy} \\ &= \sum_{n,m} F(x, n) e^{-m\theta/2\bar{\partial}_x} e^{n\theta/2\bar{\partial}_x} G(x, n) e^{i(n+m)y} = \\ &= \sum F(x - \frac{m\theta}{2}, n) G(x + \frac{n\theta}{2}, m) e^{i(n+m)y} \end{aligned} \quad (43)$$

We now recall the intuitive picture of  $F(x, n)$  as a “string” attached at  $x$  and with length  $n\theta$  along the  $x$  axis (see Fig. 8). If we regard the strings whose endpoints belong to the same leaf (i. e. are equivalent with respect to the leaf induced relation of equivalence) as matrices (since the endpoints of  $F$  and  $G$  coincide), then the Fourier transform of  $F * G$  is exactly the matrix product:

$$\sum_n F(x - \frac{k-n}{2}\theta, n) G(x + \frac{n\theta}{2}, (k-n)) \quad (44)$$

(remember  $m = k - n$ ).

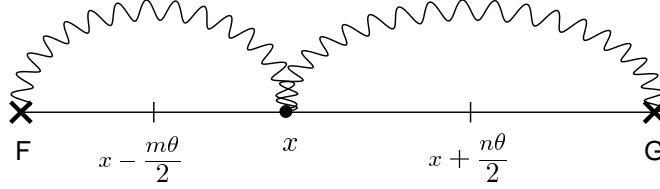
Notice that a given leaf corresponds to a point on a small circle. It was actually to the purpose of removing (this particular kind of) non locality that the small circle has been introduced.

Similarly, expressions like

$$\int (F * G)(H * K) dx dy \quad (45)$$

translate in matrix language

$$\int_0^{\frac{\Sigma}{N}} Tr (FG)(HK) \quad (46)$$

Fig. 8.  $F(x, n)$  as a “string”.

Thus, going back to the action term (41) we obtain

$$Tr \int_0^{\frac{\Sigma}{N}} [\phi^i, \phi^j]^2 dx dy \quad (47)$$

that is, our usual Yang-Mills terms.

Let us now discuss what happens if  $\theta = p/N$ ,  $p$  and  $N$  relatively prime. We have already discussed how this amounts to a “relabeling” of the sectors of the circle by means of a permutation of their indices. However, it is important to point out what happens to the “boundary condition”  $U$  and to the “Wilson loop”  $V$  matrices.

The first point to realize is that, whatever the permutation of sectors may be, the “periodic” boundary conditions will not change. This is because, no matter what happens to the interactions, the free evolution will move along the interval in the original order; the free part of the lagrangian on the big circle is certainly local.

What happens to the Wilson loop? If we permute the intervals

$$\begin{aligned} 1 &\longrightarrow 1 \\ 2 &\longrightarrow (1+p)_{\text{mod } N} \\ 3 &\longrightarrow (1+2p)_{\text{mod } N} \\ 4 &\longrightarrow (1+3p)_{\text{mod } N} \\ &\dots\dots\dots \end{aligned} \quad (48)$$

we will replace eq. (30) with

$$W(x) = \exp\left(ip \frac{x}{\Sigma}\right) V^p \quad (49)$$

The non abelian part of the Wilson loop  $V$  is replaced with

$$V^p = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \xi^{(1+p)_{\text{mod } N-1}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi^{(1+2p)_{\text{mod } N-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi^{(1+3p)_{\text{mod } N-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{pmatrix} \quad \xi = e^{2\pi i/N} \quad (50)$$

The abelian vector potential is now  $p$  times larger, and so is the flux.

Also the commutation relation is accordingly modified:

$$UW = WU \xi^p \quad (51)$$

(the rewriting in terms of the Wilson loop is allowed since the abelian part gives no contribution).

To summarize, the relations in the general case are

$$\begin{aligned} \theta &= \frac{p}{N} \\ \begin{cases} \phi(x + \frac{\Sigma}{N}, y) = U^\dagger \phi(x, y) U \\ \phi(x, y + \frac{\Sigma}{N}) = \phi(y) \end{cases} \end{aligned} \quad (52)$$

$$W = V^p$$

$$UW = WU \xi^p$$

Moreover, the abelian part of the vector potential carries now  $p$  units of magnetic flux.

### 3.4. Coupling constant rescaling

It might be interesting to derive the behaviour of the coupling constant  $g_{YM}$  along the process of “folding” the big circle into the small one. We know (cfr. for example <sup>17</sup>) that the Yang-Mills coupling for a 2+1 dim gauge theory describing compactification on a 2-torus is

$$g_{YM}^2 = \frac{1}{\Sigma} \frac{l_{11}^3}{L^3} \quad (53)$$

This is the coupling constant of the non commutative geometry on the “big” torus. The transition to the small torus yields a factor of  $1/N$ :

$$\tilde{g}_{YM}^2 = \frac{1}{N\Sigma} \frac{l_{11}^3}{L^3} \quad (54)$$

Thus  $g^2 N$  is kept fixed during the passage to the “small” torus.

### 3.5. Summary

To summarize, we have now a prescription for the link between a super Yang Mills theory with large  $N$  and a non-commutative geometry with  $\theta$  “almost irrational”.

- Take  $\theta$  and approximate it with an irreducible fraction  $\frac{p}{N}$

$\Downarrow$

- Build a  $U(N)$  gauge theory on a torus of size  $\frac{\Sigma}{N}$  and choose as boundary conditions those which are explicit in eq. (52) with  $p$  units of magnetic flux.

Thus we find that abelian gauge theory on a non-commutative torus is equivalent to an appropriate limit of non abelian gauge theory on a rescaled commutative torus.

#### 4. Wilson Lines on the Fuzzy Torus

In this section we will be interested in the construction of gauge invariant Wilson Loops in a regularized version of non-commutative gauge theory<sup>18</sup>. The theory we are using and much of our results have been discussed for the first time by Ambjorn, Makeenko, Nishimura and Szabo<sup>19</sup>. Here we give much less general, but hopefully more simple, presentation of those results.

The regularized theory is a non-commutative version of lattice gauge theory on the Fuzzy Torus. It is patterned after the Hamiltonian form of lattice gauge theory<sup>20</sup>.

The lattice version is an especially intuitive formulation of the non-perturbative theory. For illustrative purposes we will concentrate on the Abelian theory in  $2 + 1$  dimensions. The generalization to higher dimensions and non-abelian gauge groups is straightforward. Our main focus will be on defining the gauge invariant quantities of the theory including closed and open Wilson lines and in formulating the theory of matter in the fundamental representation of the non-commutative algebra of functions.

##### 4.1. Lattice Gauge theory on the Fuzzy Torus

The fuzzy torus is analogous to a periodic lattice. If we introduce coordinates

$$y = qR, \quad x = pR \quad (55)$$

such that

$$[y, x] = i\theta R^2 \quad (56)$$

and moreover choose  $\theta = 2\pi/N$ , the lattice spacing is

$$a = 2\pi R/N. \quad (57)$$

This is because the Fourier expansion in eq.(8) has only a finite number of terms. In other words there is a largest momentum in each direction

$$p_{max} = 2\pi(N - 1)/R \quad (58)$$

Thus the fuzzy torus has both an infrared cutoff length  $R$  and an ultraviolet cutoff length  $2\pi R/N$

The operators  $U, V$  function as shifts on the periodic lattice. Using the last of eq's(6) one easily finds

$$\begin{aligned} U\phi(U, V)U^\dagger &= \phi(U, Ve^{i\theta}) \\ U^\dagger\phi(U, V)U &= \phi(U, Ve^{-i\theta}) \\ V\phi(U, V)V^\dagger &= \phi(Ue^{-i\theta}, V) \end{aligned}$$

$$V^\dagger \phi(U, V) V = \phi(Ue^{i\theta}, V) \quad (59)$$

More generally

$$U^n V^m \phi(U, V) V^{\dagger m} U^{\dagger n} = \phi(Ue^{-im\theta}, Ve^{in\theta}) \quad (60)$$

The rule for integration on the fuzzy torus is simple.

$$\int U_{mn} = 4\pi^2 R^2 \delta_{m0} \delta_{n0} \quad (61)$$

Noting that

$$\text{Tr} U_{mn} = N \delta_{m0} \delta_{n0} \quad (62)$$

we make the identification

$$\int F(U, V) = \frac{4\pi^2 R^2}{N} \text{Tr} F(U, V) \quad (63)$$

In what follows we will work in the temporal gauge in which the time component of the vector potential is zero.

Let us introduce gauge fields on the fuzzy torus in analogy with the link variables of lattice gauge theory<sup>13</sup>. We will explicitly work with the gauge group  $U(1)$ . The link variable in the  $x, y$  direction is called  $X, Y$ . The link variables are unitary

$$\begin{aligned} X^\dagger X &= 1 \\ Y^\dagger Y &= 1 \end{aligned} \quad (64)$$

The gauge invariance of the theory is patterned on that of lattice gauge theory. Let  $Z$  be a unitary, time independent function of  $U, V$ ,  $Z^\dagger Z = 1$ . The gauge transformation induced by  $Z$  is defined to be

$$\begin{aligned} X' &= Z(U, V) X(U, V) Z^\dagger(Ue^{i\theta}, V) \\ Y' &= Z(U, V) Y(U, V) Z^\dagger(U, Ve^{i\theta}) \end{aligned} \quad (65)$$

or

$$\begin{aligned} X' &= Z X V^\dagger Z^\dagger V \\ Y' &= Z Y U Z^\dagger U^\dagger \end{aligned} \quad (66)$$

Let us now construct Wilson loops by analogy with the conventional lattice construction. We will give some examples first. A Wilson line which winds around the  $x$ -cycle of the torus at a fixed value of  $y$  is given by

$$\begin{aligned} W_x &= \text{Tr} X(U, V) X(Ue^{i\theta}, V) X(Ue^{2i\theta}, V) \dots X(Ue^{i(N-1)\theta}, V) \\ &= \text{Tr} (X V^\dagger)^N \end{aligned} \quad (67)$$

Similarly

$$W_y = \text{Tr} (Y U)^N \quad (68)$$

These expressions are gauge invariant under the transformation in eq.(66).

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Another example of a Wilson loop is the analogue of the plaquette in lattice gauge theory. It is given by

$$\begin{aligned}\mathcal{P} &= \text{Tr} X(U, V) Y(U e^{i\theta}, V) X^\dagger(U, V e^{i\theta}) Y^\dagger(U, V) \\ &= \text{Tr}(X)(V^\dagger Y V)(U X^\dagger U^\dagger)(Y^\dagger) \\ &= e^{-i\theta} \text{Tr}(X V^\dagger)(Y U)(V X^\dagger)(U^\dagger Y^\dagger)\end{aligned}\quad (69)$$

The general rule involves drawing a closed oriented chain formed from directed links. A step in the positive (negative)  $x$  direction is described by the link operator  $XV^\dagger$  ( $VX^\dagger$ ). Similarly a step in the positive (negative)  $y$  direction gives a factor  $YU$  ( $U^\dagger Y^\dagger$ ). The link operators are multiplied in the order specified by the chain and the trace is taken. In addition there is a factor  $e^{-iA\theta}$  where  $A$  is the signed Area of the loop in units of the lattice spacing. For a simple contractable clockwise oriented loop with no crossings,  $A$  is just the number of enclosed plaquettes.

A simple Lagrangian for the gauge theory can be formed from plaquette operators and kinetic term involving time derivatives. The expression

$$\text{Tr} \dot{X}^\dagger \dot{X} + \dot{Y}^\dagger \dot{Y} \quad (70)$$

is quadratic in time derivatives and is gauge invariant. Again, following the model of lattice gauge theory<sup>20</sup> we choose the action

$$\mathcal{L} = \frac{4\pi^2 R^2}{g^2 a^2 N} \text{Tr} \left[ \dot{X}^\dagger \dot{X} + \dot{Y}^\dagger \dot{Y} + \frac{e^{-i\theta}}{a^2} (XV^\dagger)(YU)(VX^\dagger)(U^\dagger Y^\dagger) + cc \right] \quad (71)$$

Evidently the operators  $XV^\dagger$  and  $YU$  play an important role. We therefore define

$$\begin{aligned}\mathcal{X} &= XV^\dagger \\ \mathcal{Y} &= YU\end{aligned}\quad (72)$$

These operators transform simply under gauge transformations:

$$\begin{aligned}\mathcal{X} &\rightarrow Z \mathcal{X} Z^\dagger \\ \mathcal{Y} &\rightarrow Z \mathcal{Y} Z^\dagger\end{aligned}\quad (73)$$

The action is now written in the form

$$\mathcal{L} = \frac{4\pi^2 R^2}{g^2 a^2 N} \text{Tr} \left[ \dot{\mathcal{X}}^\dagger \dot{\mathcal{X}} + \dot{\mathcal{Y}}^\dagger \dot{\mathcal{Y}} + \frac{e^{-i\theta}}{a^2} \mathcal{X} \mathcal{Y} \mathcal{X}^\dagger \mathcal{Y}^\dagger + cc \right] \quad (74)$$

or using eq.(57)

$$\mathcal{L} = \frac{N}{g^2} \text{Tr} \left[ \dot{\mathcal{X}}^\dagger \dot{\mathcal{X}} + \dot{\mathcal{Y}}^\dagger \dot{\mathcal{Y}} + \frac{e^{-i\theta}}{a^2} \mathcal{X} \mathcal{Y} \mathcal{X}^\dagger \mathcal{Y}^\dagger + cc \right] \quad (75)$$

In this form the action is equivalent to that of a  $U(N)$  lattice gauge theory formulated on a single plaquette but with periodic boundary conditions of a torus. This appears to be a form of Morita equivalence<sup>21</sup>.

If the coupling constant is small, the ground state is determined by minimizing the plaquette term in the Hamiltonian. This is done by setting

$$e^{-i\theta} \mathcal{X} \mathcal{Y} \mathcal{X}^\dagger \mathcal{Y}^\dagger = 1 \quad (76)$$

Up to a gauge transformation the unique solution of this equation is

$$\begin{aligned} \mathcal{X} &= V^\dagger \\ \mathcal{Y} &= U \end{aligned} \quad (77)$$

or

$$X = Y = 1 \quad (78)$$

#### 4.2. Open Wilson Loops

Thus far we have constructed closed Wilson loops. Recall that the construction involves taking a trace. This is the analogue of integrating the location of the Wilson loop over all space. In other words the closed Wilson Loop carries no spatial momentum. In a very interesting paper Ishibashi, Iso, Kawai and Kitazawa<sup>22</sup> have argued that there exist gauge invariant operators which correspond to specific Fourier modes of open Wilson lines. These objects are very closely related to the growing dipoles of non-commutative field theory whose size depends on their momentum<sup>23,24,25</sup>. Das and Rey<sup>26</sup> have shown that these operators are a complete set of gauge invariant operators. Their importance has been further clarified by Gross, Hashimoto and Itzhaki<sup>27</sup>.

Let us consider the simplest example of an open Wilson line, ie, a single link variable, say  $X$ . From eq.(66) we see that  $X$  is not gauge invariant. But now consider  $XV^\dagger = \mathcal{X}$ . Under gauge transformations

$$\mathcal{X} \rightarrow Z \mathcal{X} Z^\dagger \quad (79)$$

Evidently the quantity

$$Tr X V^\dagger = Tr \mathcal{X} \quad (80)$$

is gauge invariant. Now using (3) and (55) we identify this quantity as

$$Tr X V^\dagger = \frac{N}{4\pi^2 R^2} \int X e^{\frac{-iy}{R}} d^2 x \quad (81)$$

Thus we see that a particular Fourier mode of  $X$  is gauge invariant.

Let us consider another example in which an open Wilson line consist of two adjacent links, one along the  $x$  axis and one along the  $y$  axis.

$$X V^\dagger Y V = \mathcal{X} \mathcal{Y} U^\dagger V \quad (82)$$

Multiplying by  $V^\dagger U$  and taking the trace gives

$$Tr(X V^\dagger Y V) V^\dagger U = Tr \mathcal{X} \mathcal{Y} = \text{gauge invariant} \quad (83)$$

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But we can also write this as

$$\frac{N}{4\pi^2 R^2} \int d^2x (XV^\dagger YV) e^{\frac{-iy}{R}} e^{\frac{-ix}{R}} \quad (84)$$

In other words it is again a Fourier mode of the open Wilson line. In general the particular Fourier mode is related to the separation between the endpoints of the Wilson line by the same relation as that in <sup>24,25</sup> where it was shown that a particle in non-commutative field theory is a dipole oriented perpendicular to its momentum with a size proportional to the momentum.

#### 4.3. *Fields in the Fundamental Representation*

In this section we will define fields in the fundamental representation of the gauge group. For simplicity we consider non-relativistic particles. Let us begin with what we do *not* mean by particles in the fundamental. Define a complex valued field  $\phi$  that takes values in the  $N \times N$  dimensional matrix algebra generated by  $U, V$ . The gauge transformation properties of  $\phi$  are given by

$$\phi \rightarrow Z\phi. \quad (85)$$

Note that this is left multiplication by  $Z$  and not conjugation. The field  $\phi$  carries a single unit of abelian gauge charge. Although the field has two indices in the  $N$  dimensional space the gauge transformations only act on the left index.

An obvious choice of gauge invariant "hopping" Hamiltonian would be

$$H \sim \text{Tr} \phi^\dagger X V^\dagger \phi V \phi^\dagger X U \phi U^\dagger + cc \quad (86)$$

In a non-abelian theory a similar construction can be carried out for quark fields in the fundamental.

We shall mean something different by fields in the fundamental. Such fields have only *one* index. They are vectors rather than matrices in the Hilbert space that they represent the algebra of functions. In the present case they are  $N$  component complex vectors  $|\psi\rangle$ . These fields represent particles moving in a strong magnetic field which are frozen into the lowest Landau level.

Consider the case of non-relativistic particles moving on the non-commutative lattice. The conventional lattice action would be

$$L = L_0 - L_h \quad (87)$$

where

$$L_0 = i \left( \langle \psi^\dagger | \psi \rangle - cc \right) \quad (88)$$

and  $L_h$  is a hopping Hamiltonian. The natural non-commutative version of the hopping term is

$$L_h = \frac{1}{a} \langle \psi | X V^\dagger + Y U - 2 | \psi \rangle + cc \quad (89)$$

The presence of the link variables  $X, Y$  is familiar from ordinary lattice field theory and the  $V^\dagger, U$  are the shifts which move  $\psi$ . We may also write the hopping term as

$$L_h = \frac{1}{a} \langle \psi | \mathcal{X} + \mathcal{Y} | - 2\psi \rangle + cc \quad (90)$$

Combining (75), (86) and (88)

$$\mathcal{L} = \frac{N}{g^2} Tr \left[ \dot{\mathcal{X}}^\dagger \dot{\mathcal{X}} + \dot{\mathcal{Y}}^\dagger \dot{\mathcal{Y}} + \frac{e^{-i\theta}}{a^2} \mathcal{X} \mathcal{Y} \mathcal{X}^\dagger \mathcal{Y}^\dagger + cc \right] + i \langle \dot{\psi}^\dagger | \psi \rangle + \frac{1}{a} \langle \psi | \mathcal{X} + \mathcal{Y} - 2|\psi \rangle + cc \quad (91)$$

Let us consider hopping terms in (90). In the limit of weak coupling we may use eq.(77) to give

$$L_h = \frac{1}{a} \langle \psi | V^\dagger + U - 2|\psi \rangle + cc \quad (92)$$

To get some idea of the meaning of this term let us use eqs.(3)-(55) and expand the exponentials.

$$\begin{aligned} L_h &= \frac{1}{a} \langle \psi | \frac{(x^2 + y^2)}{R^2} | \psi \rangle + cc = \\ &= \frac{1}{a} \langle \psi | (p^2 + q^2) \theta | \psi \rangle + cc \end{aligned} \quad (93)$$

As  $[p, q] = i\theta$ , we recognize this term as a harmonic oscillator hamiltonian with an in spectrum of levels spaced by  $\theta \sim N^{-1}$ . Evidently, in this approximation the particles move in quantized circular orbits around the origin.

This phenomena is related to the fact that the fundamental particles behave like charged particles in a strong magnetic field and are frozen into their lowest Landau levels. Furthermore the LLL's are split by a force attracting the particles to  $x = y = 0$ . This has a natural interpretation in matrix theory in which the same system appears as a 2-brane and 0-brane with strings connecting them <sup>28</sup>.

#### 4.4. Rational Theta

Thus far we have worked with eq.(6) with  $\theta = 2\pi/N$ . Let us generalize the construction to the case  $\theta = 2\pi p/N$  with  $p$  relatively prime to  $N$ . We continue to define the fuzzy torus by eq.(6). Let us define two matrices  $u, v$  satisfying

$$\begin{aligned} u^\dagger u &= v^\dagger v = 1 \\ u^N &= v^N = 1 \\ uv &= v u e^{\frac{2\pi i p}{N}} \end{aligned} \quad (94)$$

where

$$\alpha p = 1 \pmod{N} \quad (95)$$

Then it follows that

$$U = u^p$$

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$$V = v^p \quad (96)$$

satisfies eq.(6). Furthermore,  $u$  and  $v^\dagger$  act as shifts by distance  $2\pi R/N$ ;

$$\begin{aligned} uVu^\dagger &= V \exp\left(\frac{2\pi i}{N}\right) \\ v^\dagger Uv &= U \exp\left(\frac{2\pi i}{N}\right) \end{aligned} \quad (97)$$

The basic plaquette is now given by

$$\begin{aligned} \mathcal{P} &= \text{Tr}(X)(v^\dagger Yv)(uX^\dagger u^\dagger)(Y^\dagger) \\ &= e^{\frac{2\pi i\alpha}{N}} \text{Tr}(Xv^\dagger)(Yu)(vX^\dagger)(u^\dagger Y^\dagger) \end{aligned} \quad (98)$$

The final expression for action is essentially the same as in eq.(69) except that the factor  $e^{i\theta}$  is replaced by  $e^{\frac{2\pi i\alpha}{N}}$ .

We can now describe one approach to the continuum limit,  $a \rightarrow 0$ . To get to such a limit eq.(57) requires that  $N \rightarrow \infty$ . We also want the theta parameter to approach a finite limit. This requires  $p/N$  to approach a limit. for example if we want  $p/N \rightarrow 1/2$  we can choose the sequence ( $p = n, N = 2n + 1$ ) so that  $p$  and  $N$  remain relatively prime. In this way  $p/N$  can tend to a rational or irrational limit and the lattice spacing will approach zero.

## 5. String Theory and Non-commutative Geometry

We finally give a simple physical picture of how non-commutative geometry arises in string theory. As we have discussed in the previous sections, the structure of such theories is similar to that of ordinary gauge theory except that the usual product of fields is replaced by a “star product” defined by

$$\phi * \chi = \phi(X) \exp\{-i\theta^{\mu\nu} \frac{\partial}{\partial X^\mu} \frac{\partial}{\partial Y^\nu}\} \chi(Y) \quad (99)$$

where  $\theta^{\mu\nu}$  is an antisymmetric constant tensor. The effect of such a modification is reflected in the momentum space vertices of the theory by factors of the form

$$\exp[i\theta^{\mu\nu} p_\mu q_\nu] \equiv e^{ip \wedge q} \quad (100)$$

The purpose of this section is to show how these factors arise in an elementary way. We will begin by describing a simple quantum mechanical system which is fundamental to our construction. We then consider string theory in the presence of a D3-brane and a constant large  $B_{\mu\nu}$  field. In the light cone frame the first quantized string is described by our elementary model. We use the model to compute the string splitting vertex and show how the factors in eq. (100) emerge. We then turn to the structure of the perturbation series for the non-commutative theory in infinite flat space. We find that planar diagrams with any number of loops are identical to their commutative counterparts apart from trivial external line phase factors. The effects of non-commutativity in a finite size geometry, as the fuzzy torus of the previous sections, are more subtle. We comment on this issue towards the end of this section.

### 5.1. The model

#### 5.1.1. Classical level

Consider a pair of unit charges of opposite sign in a magnetic field  $B$  in the regime where the Coulomb and the radiation terms are negligible. The coordinates of the charges are  $\vec{x}_1$  and  $\vec{x}_2$  or in component form  $x_1^i$  and  $x_2^i$ . The Lagrangian is

$$\mathcal{L} = \frac{m}{2} ((\dot{x}_1)^2 + (\dot{x}_2)^2) + \frac{B}{2} \epsilon_{ij} (\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j) - \frac{K}{2} (x_1 - x_2)^2 \quad (101)$$

where the first term is the kinetic energy of the charges, the second term is their interaction with the magnetic field and the last term is an harmonic potential between the charges.

In what follows we will be interested in the limit in which the first term can be ignored. This is typically the case if  $B$  is so large that the available energy is insufficient to excite higher Landau levels<sup>29</sup>. Thus we will focus on the simplified Lagrangian

$$\mathcal{L} = \frac{B}{2} \epsilon_{ij} (\dot{x}_1^i x_1^j - \dot{x}_2^i x_2^j) - \frac{K}{2} (x_1 - x_2)^2 \quad (102)$$

Let us first discuss the classical system. In terms of the center of mass and relative coordinates  $X$ ,  $\Delta$ :

$$\begin{aligned} \vec{X} &= (\vec{x}_1 + \vec{x}_2)/2 \\ \vec{\Delta} &= (\vec{x}_1 - \vec{x}_2)/2 \end{aligned} \quad (103)$$

the Lagrangian is

$$\mathcal{L} = m((\dot{X})^2 + (\dot{\Delta})^2) + 2B\epsilon_{ij}\dot{X}^i\Delta^j - 2K(\Delta)^2 \quad (104)$$

Dropping the kinetic terms gives

$$\mathcal{L} = 2B\epsilon_{ij}\dot{X}^i\Delta^j - 2K(\Delta)^2 \quad (105)$$

The momentum conjugate to  $X$  is

$$\frac{\partial \mathcal{L}}{\partial \dot{X}^i} = 2B\epsilon_{ij}\Delta^j = P_i \quad (106)$$

This is the center of mass momentum.

Finally, the Hamiltonian is

$$\mathcal{H} = 2K(\Delta)^2 = 2K \left( \frac{P}{2B} \right)^2 = \frac{K}{2B^2} P^2 \quad (107)$$

This is the hamiltonian of a nonrelativistic particle with mass

$$M = \frac{B^2}{K} \quad (108)$$

Evidently the composite system of opposite charges moves like a galileian particle of mass  $M$ . What is unusual is that the spatial extension  $\Delta$  of the system is related to its momentum so that the size grows linearly with  $P$  according to

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eq. (106). How does this growth with momentum effect the interactions of the composite? Let's suppose charge 1 interacts locally with an "impurity" centered at the origin. The interaction has the form

$$V(\vec{x}_1) = \lambda \delta(\vec{x}_1) \quad (109)$$

In terms of  $X$  and  $\Delta$  this becomes

$$V = \lambda \delta(X + \Delta) = \lambda \delta(X^i - \frac{1}{2B} \epsilon^{ij} P_j) \quad (110)$$

Note that the interaction in terms of the center of mass coordinate is nonlocal in a particular way. The interaction point is shifted by a momentum dependent amount. This is the origin of the peculiar momentum dependent phases that appear in interaction vertices on the non-commutative plane. More generally, if particle 1 sees a potential  $V(x_1)$  the interaction becomes

$$V\left(X - \frac{\epsilon P}{2B}\right) \quad (111)$$

### 5.1.2. *Quantum level*

The main problem in quantizing the system is to correctly define expressions like (111) which in general have factor ordering and other quantum ambiguities. A standard way of solving this problem is Weyl ordering: assume that  $V$  can be expressed as a Fourier transform

$$V(x) = \int dq \tilde{V}(q) e^{iqx} \quad (112)$$

We can then formally write

$$V(X - \frac{\epsilon P}{2B}) = \int dq \tilde{V}(q) e^{iq(X - \frac{\epsilon P}{2B})} \quad (113)$$

where the factor ordering in the exponential is not ambiguous since:

$$[q_i X^i, q_l \epsilon^{lj} P_j] = q_i q_l \epsilon^{lj} [X^i P_j] = 0 \quad (114)$$

Let  $\langle k|$  and  $|l\rangle$  be momentum eigenvectors. The commutation relations (114) immediately imply that

$$\langle k| \exp[iq(X - \frac{\epsilon P}{2B})] |l\rangle = \delta(k - q - l) \exp[-iq\epsilon l/2B] \quad (115)$$

The phase factor above is the usual Moyal bracket phase that is ubiquitous in non-commutative geometry.

### 5.2. String theory in magnetic fields

Let us consider bosonic string theory in the presence of a D3-brane. The coordinates of the brane are  $x^0, x^1, x^2, x^3$ . The remaining coordinates will play no role. We will also assume a background antisymmetric tensor field  $B_{\mu\nu}$  in the 1,2 direction. We will study the open string sector with string ends attached to the D3-brane in the light cone frame.

Define

$$x^\pm = x^0 \pm x^3 \quad (116)$$

and make the usual light cone choice of world sheet time

$$\tau = x^+ \quad (117)$$

The string action is

$$\mathcal{L} = \frac{1}{2} \int_{-L}^L d\tau d\sigma \left[ \left( \frac{\partial x^i}{\partial \tau} \right)^2 - \left( \frac{\partial x^i}{\partial \sigma} \right)^2 + B_{ij} \left( \frac{\partial x^i}{\partial \tau} \right) \left( \frac{\partial x^j}{\partial \sigma} \right) \right] \quad (118)$$

We have numerically fixed  $\alpha'$  and the parameter  $L$  can be identified with  $P_-$ , the momentum conjugate to  $x_-$ .

In what follows we will be interested in deriving the limit  $B \rightarrow \infty$  of the action above, by keeping fixed the following rescaled variables

$$\begin{cases} x^i = \frac{y^i}{\sqrt{B}} \\ \tau = tB \end{cases} \quad (119)$$

After dropping a term which vanishes in the large  $B$  limit and an integration by parts, the action reduces to

$$\mathcal{L} = \frac{1}{2} \int d\sigma d\tau \left( \frac{\partial y}{\partial \sigma} \right)^2 + \epsilon_{ij} \dot{y}_i y_j \Big|_{-L}^L \quad (120)$$

Since for  $\sigma \neq \pm L$  the time derivatives of  $y$  do not appear in  $S$  we may trivially integrate them out. The solution of the classical equation of motion is

$$y(\sigma) = y + \frac{\Delta \sigma}{L} \quad (121)$$

with  $\Delta$  and  $y$  independent of  $\sigma$ . The resulting action is

$$\mathcal{L} = \left[ -\frac{2\Delta^2}{L} + \dot{y}\epsilon\Delta \right] \quad (122)$$

Evidently, the action is of the same form as the model in section 1 with  $B$  and  $K$  rescaled.

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### 5.2.1. The interaction vertex

Interactions in light cone string theory are represented by string splitting and joining. Consider two incoming strings with momenta  $p_1, p_2$  and center of mass positions  $y_1, y_2$ . If their endpoints coincide they can join to form a third string with momentum  $-p_3$ . The constraints on the endpoints are summarized by the overlap  $\delta$  function

$$\mathcal{V} = \delta((y_1 - \Delta_1) - (y_2 + \Delta_2)) \delta((y_2 - \Delta_2) - (y_3 + \Delta_3)) \delta((y_3 - \Delta_3) - (y_1 + \Delta_1)) \quad (123)$$

From eq. (122) we see that the center of mass momentum is related to  $\Delta$  by

$$P = \epsilon \Delta \quad (124)$$

Inserting this in eq. (123) gives the vertex

$$\mathcal{V} = \delta(y_1 - y_2 + (\epsilon p_1 + \epsilon p_2)) \delta(y_2 - y_3 + (\epsilon p_2 + \epsilon p_3)) \delta(y_3 - y_1 + (\epsilon p_3 + \epsilon p_1)) \quad (125)$$

To get the vertex in momentum space multiply by  $e^{i(p_1 y_1 + p_2 y_2 + p_3 y_3)}$  and integrate over  $y$ . This yields

$$\mathcal{V} = e^{i(p_1 \epsilon p_2)} \delta(p_1 + p_2 + p_3) \quad (126)$$

This is the usual form of the vertex in non-commutative field theory. We have scaled the “transverse” coordinates  $x^1, x^2$  (but not  $x^0, x^3$ ) and momenta so that the  $B$  field does not appear in the vertex. If we go back to the original units the phases in (126) will be proportional to  $1/B$ .

Evidently a quantum of non-commutative Yang Mills theory may be thought of as a straight string connecting two opposite charges. The separation vector  $\Delta$  is perpendicular to the direction of motion  $P$ .

Now consider the geometry of the 3-body vertex. The string endpoints  $u, v, w$  define a triangle with sides

$$\begin{aligned} \Delta_1 &= (u - v) \\ \Delta_2 &= (v - w) \\ \Delta_3 &= (w - u) \end{aligned} \quad (127)$$

and the three momenta are perpendicular to the corresponding  $\Delta$ . It is straightforward to see that the phase

$$\epsilon_{ij} p_i q_j / B \equiv p \wedge q \quad (128)$$

is just the area of the triangle times  $B$ . In other words, it is the magnetic flux through the triangle. Note that it can be of either sign.

More generally, we may consider a Feynman tree diagram constructed from such vertices. For example consider figure (9a). The overall phase is the total flux through the triangles A, B and C. In fact we can simplify this by shrinking the internal propagators to get figure (9b). Thus the phase is the flux through a polygon

formed from the  $\Delta$ 's of the external lines. The phase depends only on the momenta of the external lines and their cyclic order.

### 5.3. Structure of Perturbation Theory

In this section we will consider the effects of the Moyal phases on the structure of Feynman amplitudes in non-commutative Yang Mills theory. Let us first review the diagram rules for ordinary Yang Mills theory in 't Hooft double-line representation.

The gauge propagator can be represented as a double line as if the gauge boson were a quark-antiquark pair as in figure (10). Each gluon is equipped with a pair of gauge indices  $i, j$ , a momentum  $p$  and a polarization  $\varepsilon$  satisfying  $\varepsilon \cdot p = \varepsilon^\mu p_\mu = 0$ .

The vertex describing 3-gauge boson interaction is shown in figure (11). In addition to Kronecker  $\delta$  for the gauge indices and momentum  $\delta$  functions the vertex contains the factor

$$(\varepsilon_1 \cdot p_3 + \varepsilon_3 \cdot p_2 + \varepsilon_2 \cdot p_1) \quad (129)$$

The factor is antisymmetric under interchange of any pair and so it must be accompanied by an antisymmetric function of the gauge indices. For a purely abelian theory the vertex vanishes when symmetrized.

Now we add the new factor coming from the Moyal bracket. This factor is

$$e^{ip_1 \wedge p_2} = e^{ip_2 \wedge p_3} = e^{ip_3 \wedge p_1} \quad (130)$$

where  $p_a \wedge p_b$  indicates an antisymmetric product

$$\begin{aligned} p \wedge q &= p_\mu q_\nu \theta^{\mu\nu} \\ \theta^{\mu\nu} &= -\theta^{\nu\mu} \end{aligned} \quad (131)$$

Because these factors are not symmetric under interchange of particles, the vertex no longer vanishes when Bose symmetrized even for the Abelian theory.

The phase factors satisfy an important identity. Let us consider the phase factors that accompany a given diagram. In fact from now on a diagram will indicate **only the phase factor** from the product of vertices. Now consider the diagram in figure (12a). It is given by

$$e^{i(p_1 \wedge p_2)} e^{i(p_1 + p_2) \wedge p_3} = e^{i(p_1 \wedge p_2 + p_2 \wedge p_3 + p_1 \wedge p_3)} \quad (132)$$

On the other hand the dual diagram figure (12b) is given by  $e^{i(p_1 \wedge (p_2 + p_3) + p_2 \wedge p_3)}$ . It is identical to the previous diagram. Thus the Moyal phases satisfy old fashioned “channel duality”. This conclusion is also obvious from the “flux through polygon” construction of the previous section.

In what follows, a “duality move” will refer to a replacement of a diagram such as in figure (12a) by the dual diagram in figure (12b).

Now consider any planar diagram with  $L$  loops. By a series of “duality moves” it can be brought to the form indicated in figure (13) consisting of a tree with  $L$  simple one-loop tadpoles.

Let us consider the tadpole, figure (14). The phase factor is just  $e^{iq\wedge q} = 1$ . Thus the loop contributes nothing to the phase and the net effect of the Moyal factors is exactly that of the tree diagram. In fact all trees contributing to a given topology have the same phase, which is a function only of the external momentum. The result is that for planar diagrams the Moyal phases do not affect the Feynman integrations at all. In particular the planar diagrams have exactly the same divergences as in the commutative theory. Evidently in the large N limit non-commutative field theory = ordinary field theory<sup>d</sup>.

On the other hand, divergences that occur in nonplanar diagrams are generically regulated by the phase factors. For example consider the nonplanar diagram in figure (15). The Moyal phase for the diagram is

$$e^{ip\wedge q}e^{ip\wedge q} = e^{2ip\wedge q} \quad (133)$$

where  $p$  is the external momentum and  $q$  the momentum circulating in the loop. It is not difficult to see that such oscillating phases will regulate divergent diagrams and make them finite, unless the diagram contains divergent planar subdiagrams. However there is an exception to the rule that nonplanar diagrams are finite. If a line with a nonplanar self energy insertion such as in figure (15) happens to have vanishing momentum in the 1,2 plane then according to eq. (133) its phase will vanish. Thus it seems that the leading high momentum behavior of the theory is controlled by the planar diagrams and by nonplanar diagrams at exceptional values of the external momenta<sup>31</sup>.

An interesting question arises if we study the theory on a torus of finite size<sup>21</sup>. For an ordinary local field theory high momentum behavior basically corresponds to small distance behavior. For this reason we expect the high momentum behavior on a torus to be identical to that in infinite space once the momentum becomes much larger than the inverse size of the torus. However in the non-commutative case the story is more interesting. We have seen that high momentum in the 1,2 plane is associated with *large* distances in the perpendicular direction. Most likely this means that the finite torus generically behaves very differently at high momentum than the infinite plane. Indeed, as we noticed above, nonplanar diagrams diverge for exceptional values of the external momenta. Since the set of exceptional momenta is of zero measure, this presumably leads to no divergence in the self-energy in infinite space when the external momenta in question are integrated over. The situation could be different for compact non-commutative geometries since integrals over momenta are replaced by sums<sup>32</sup>.

The fact that the large N limit is essentially the same for non-commutative and ordinary Yang Mills theories implies that in the AdS/CFT correspondence the introduction of non-commutative geometry does not change the thermodynamics

<sup>d</sup>In the case of Yang-Mills theory subtleties can arise: for example the non-commutative  $U(1)$  gauge theory is a non-trivial, non-free theory even in the planar limit and, thus, not equivalent to its commutative counterpart<sup>30</sup>.

of the theory<sup>33</sup>. It may also be connected to the fact that in the matrix theory construction of Connes-Douglas-Schwartz and Douglas-Hull, the large  $N$  limit effectively decompactifies  $X^{11}$  and should therefore eliminate dependence on the 3-form potential. However the argument is not straightforward since in matrix theory we are not usually in the 't Hooft limit.

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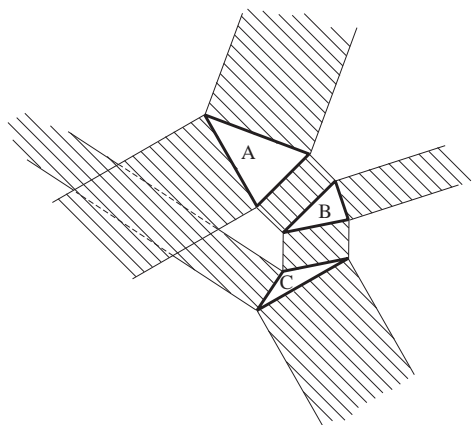


Fig. 9a

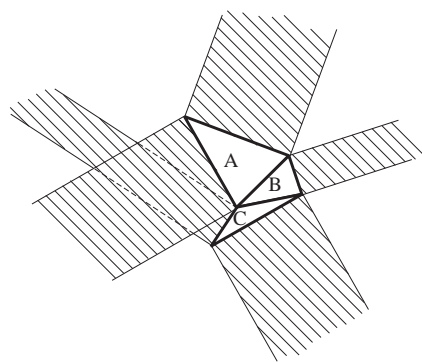
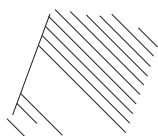


Fig. 9b

"Feynman tree diagram" for the scattering of strings.



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Fig. 10

Double line representation of the propagator.

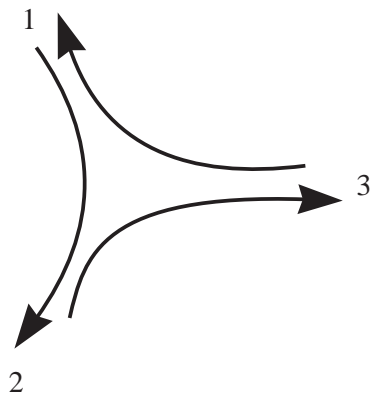


Fig. 11a

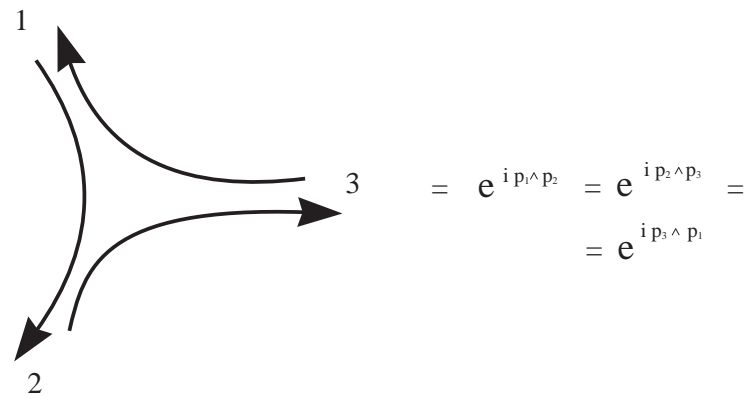


Fig. 11b

The three-boson interaction and its associated phase.

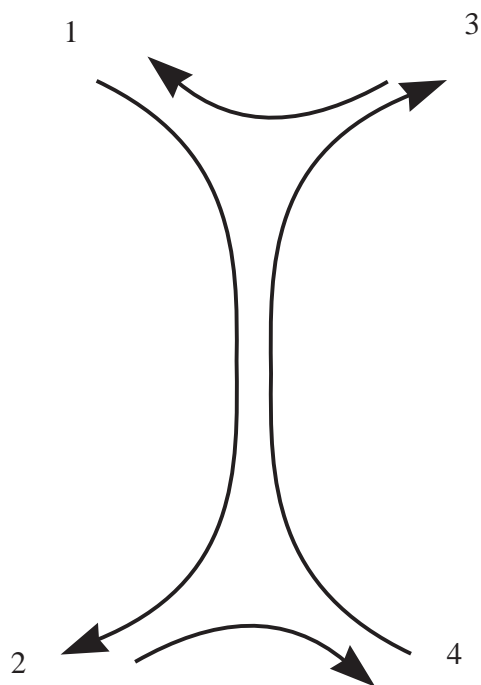


Fig. 12a

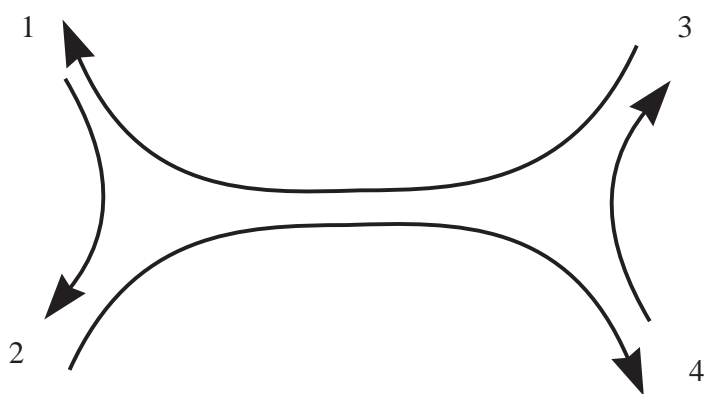


Fig. 12b

The elementary "duality exchange" move.

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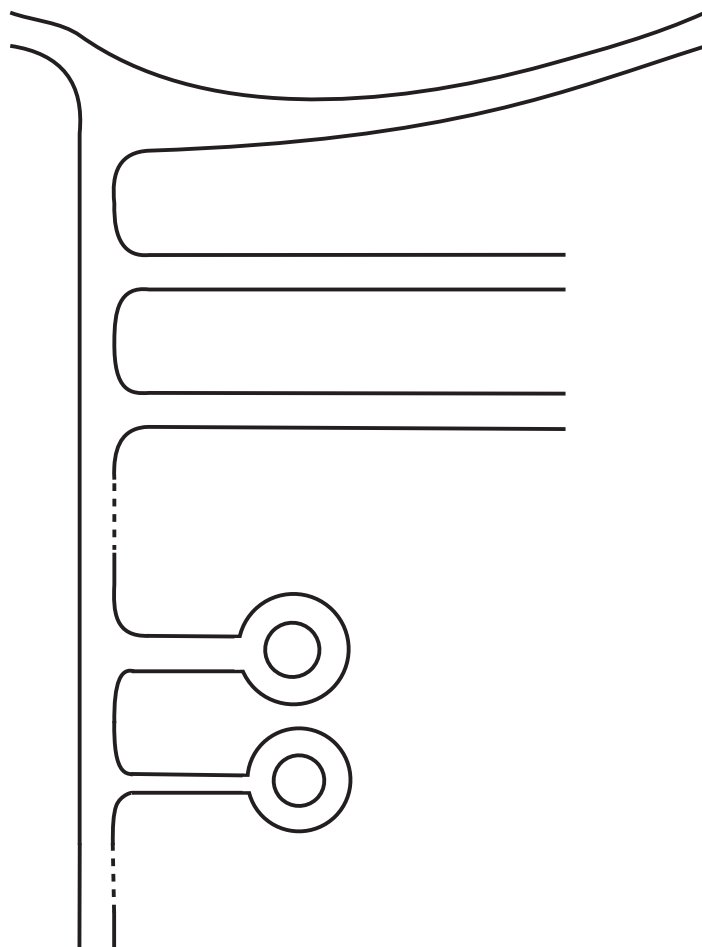


Fig. 13

Basic form of any planar diagram after appropriate sequence of duality moves.

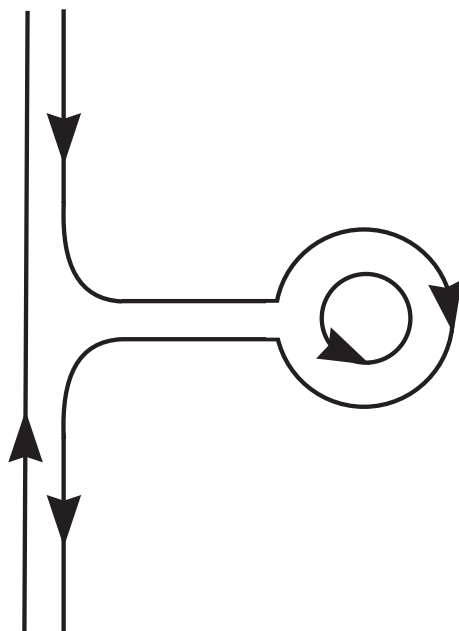


Fig. 14

The tadpole diagram.



Fig. 15

Non planar insertion for the self-energy.